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$SU(3)$ trits of orbifolded $E_8 \times E_8'$ heterotic string and supersymmetric standard model

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Abstract: We present Z_3 orbifold compactifications of $E_8 \times E_8'$ heterotic string with three Wilson lines, resulting to the maximum number of $SU(3)$ factors. Here, all the matter spectrums are in the $SU(3)$ trits($\equiv$ three representations $\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}$) of the $SU(3)^8$ GUT. Using this information, we show how three family supersymmetric standard models(SSM) can be obtained. Also, the low lying interesting representations(fundamental and adjoint) of E_6 and E_8 are given in terms of trits, establishing simple criteria for treating these low lying representations of exceptional groups.

Keywords: Supersymmetric SM, $SU(3)^8$ GUT, Z_3 orbifold.

Contents

1. Introduction

It is of utmost importance in string phenomenology to obtain a supersymmetric standard model(SSM) from compactifications of 10 dimensional(10D) string theory. In this regards, it was emphasized recently that a grand unification(GUT) direction, toward the electroweak hypercharge embedded in semi-simple groups without adjoint representation(HESSNA) is preferred[1]. The main arguments for semi-simple groups are to obtain easily a GUT model with the bare value of $\sin^2\theta_W^0 = \frac{3}{8}$ and the matter

spectrum needed for the GUT symmetry breaking. On the other hand, with a GUT in a simple group, one needs an adjoint representation to break it into the standard model(SM), which is difficult to obtain in the orbifold compactifications of the heterotic string[2].

Initially, the construction of *standard-like* models, using Wilson lines[3], was considered to be desirable in the hope of obtaining a SSM directly from compactification of a 10D superstring with a possibility of resolving the doublet-triplet splitting problem in GUT models[4]. If we have succeeded in the construction of a 4D SSM, it might have given a great confidance in high energy predictions of the string theory. However, we have stopped at the *standard-like* models where only the correct gauge groups and desirable matter spectrum were obtained. One notable merit in the construction of the standard-like models was that we do not need big representations to break a huge GUT group.

In these standard-like models, however, there are three theoretical problems: (i) the bare value of $\sin^2\theta_W^0$ is generally different from $\frac{3}{8}$, (ii) there appear too many Higgs doublets, and (iii) there are too many $U(1)$'s. In the resulting 4D supersymmetric gauge theory framework, (ii) and (iii) can be understood, if not solved, by the existing idea in grand unification models. At high energy, it is a natural phenomenon that vectorlike representations are removed.[1] Under this strategy, one can remove a lot of Higgs doublets except one pair of doublets for the minimal supersymmetric standard model(MSSM). But the orbifold compactification is usually too much chiral, implying that there remain too many Higgs doublets which do not form vectorlike representations due to the extra unbroken $U(1)$'s. If all the $U(1)$'s are broken except that of the electroweak hypercharge, then there is a chance that they form vectorlike representations. This happens for the case with $\sin^2\theta_W^0 = \frac{3}{8}$. By the vacuum expectation values of $U(1)$-charge carrying singlets, one can break some of the left-over $U(1)$'s. However, here one has to be careful not to break supersymmetry by the Fayet-Iliopoulos D-term[6], even though the verification of the survival of the electroweak hypercharge is time-consuming[7] and sometimes the supersymmetric vacuum is not realized. Thus, the Higgs doublet problem is also related to the $\sin^2\theta_W^0$ problem of the orbifold compactifications.[2] This $\sin^2\theta_W^0$ problem is inherent in models with extra $U(1)$'s and it cannot be simply resolved by the existing GUT idea.

This has led to simple groups[8] and flipped $SU(5)$ models[9], which was worked out in the fermionic construction. In the orbifold compactification, the $U(1)$ problem is difficult to circumvent, which is the reason that it is better to consider HESSNA in orbifold compactification[1].

For HESSNA, the most famous example is the $SU(3)^3$ group, which is sometimes

[1]This is one reason that the μ problem has turned up to be a difficult hierarchy problem[5].

[2]In fermionic constructions, it has been claimed that $\sin^2\theta_W^0$ can be $\frac{3}{8}$, but here we concentrate on the orbifold contstruction which can be viewed in terms of geometry.

called ‘trinification’[10]. Since it is a factor group, it may not be considered as a grand unification, but the trinification idea is very similar to E_6 grand unification as far as the multiplet **27** is concerned,

$$(\bar{\mathbf{3}},\mathbf{3},\mathbf{1})+(\mathbf{1},\bar{\mathbf{3}},\mathbf{3})+(\mathbf{3},\mathbf{1},\bar{\mathbf{3}}) \tag{1.1}$$

Recently, it was shown that the trinification spectrum can be obtained from the orbifold compactification [11].

For HESSNA, the factor groups $SU(3)$’s play a key role. In the heterotic string models, they are related to the exceptional group E_8. In Eq. (1.1) each $SU(3)$ has three kinds of representations, $\mathbf{3},\bar{\mathbf{3}}$, and $\mathbf{1}$, which can play an important role in the search of a SSM. In the Dynkin diagram technique, these $SU(3)$’s can be clearly seen [12, 13]. In Ref. [1], $SU(3)^4$ was obtained from a shift vector and a Wilson line, not leaking to the other E_8. This observation is very useful in finding out the maximum number of $SU(3)$ factors from the heterotic string theory. Namely, the heterotic string based on the rank–16 $E_8\times E_8'$[14] can contain eight $SU(3)$ factor groups as its maximum number. All the representations we obtain in $SU(3)$’s are $\mathbf{3},\bar{\mathbf{3}}$, and $\mathbf{1}$ which is called *the* $SU(3)$ *trits*.[3]

From the symmetry point of view, it is most interesting to consider eight $SU(3)$ factors with the trits system. To obtain these trits in the orbifold compactification, we must break $E_8\times E_8'$ with three Wilson lines.

The compactification with three Wilson lines can be a draw-back toward introducing three families, since the multiplicity of the fields is only 1 at each fixed point due to the different condition at each different fixed point. But this highly broken gauge group with $SU(3)$ trits is very useful because one obtains a complete vacuum structure in case of the orbifolding. Starting from this vacuum structure, one can enlarge the symmetry by removing a Wilson line(s). In this paper, we adopt this maximum information strategy, which is contrary to the standard method of orbifolding with fewer Wilson lines. But, when we search the matter spectrum (1.1), we use only a part of the information from the three Wilson line models. By removing one Wilson line, the GUT group can be enhanced to E_6 from $SU(3)^3$ as our constructions will show later. But it is known that the rank–6 E_6 group cannot be broken down to the rank–4 SM gauge group by the vacuum expectation values(VEV) of two independent directions of **27**. To break it down to the SM, one needs an adjoint representation. We may speculate that the heavy Kaluza-Klein modes of the internal gauge bosons provide the needed adjoint.

In this paper, we basically deal with the group theory properties of the maximally symmetric $SU(3)$ subgroups of $E_8\times E_8'$, in terms of the trits system.

[3]The binary system $\{1,0\}$ defines bits which is closed under addition mod. 2. Our set $\{\mathbf{3},\bar{\mathbf{3}},\mathbf{1}\}$ is a triple system closed under group multiplication with projecting out symmetric multiples. So, we call this set a trit.

In Sec. II, we present two schemes for the $SU(3)^8$ realization with three Wilson lines. Model A does not contain bulk matter and Model B contains bulk matter. In Sec. III, we construct a SSM from the $SU(3)$ trits of Model A. In Sec. IV, we discuss the spontaneous symmetry breaking and related issues in SSM-I. In Sec. V, we present trits algebra for an easy treatment of low lying representations of exceptional groups. In Sec. VI, we propose a mechanism for the doublet-triplet splitting. Sec. VII is a conclusion.

2. $SU(3)^8$ GUT with three Wilson lines

In this section, we present two models with $SU(3)^8$ GUT groups. The Tables we present here can be used in finding a desired HESSNA with three families, as we show in the subsequent section. These Tables show the maximally symmetric $SU(3)$ trits. In obtaining the $SU(3)$ trits, the knowledge of the shift vector and the Wilson lines of Ref. [1] is used as the building blocks.

There are reasons preferring Z_3 orbifolds. One is that Z_3 orbifolds leave a 4D $N = 1$ supersymmetry unbroken [2]. Another reason is that there appear three fixed points on a two-torus orbifolded by Z_3. In this respect, other orbifolds cannot compete with Z_3 which guarantees the multiples of 3. In the untwisted sector, the multiplicity is 3, because the Z_3 oscillator provides three cases for the chiral matter in the bulk. In addition, there is the simplicity in treating the partition functions in the Z_3 orbifolds, mainly because 3 being a prime number. The seemingly simpler Z_2 orbifold is in fact more complicated than Z_3, since it needs an extra work in compactifying 6D down to 4D, and also in figuring out the degeneracy factor in the Z_2 case [15]. Thus, the compactification of the six internal dimensions through three two-tori gives 27 fixed points. If we only use the shift vector v, then these 27 fixed points are the same in every aspect. Thus, if a particle(or a string) sits on a fixed point, it appears in the same way at each fixed point, giving the multiplicity 27. Introducing one Wilson line reduces the multiplicity by a factor of 3 in the twisted sector. If one want to distinguish every fixed point, then three Wilson lines are needed. In this way, one obtains the maximum information about the vacuum. Below, we present two such models, allowing eight $SU(3)$ trits. For the definition of $\mathbf{3}$ and $\overline{\mathbf{3}}$ of four $SU(3)$'s from one E_8, we present their E_8 root vectors in Table 1 [1].

2.1 Model A

Recently, it has been known how to extend the Kac-Peterson method [12] to include Wilson lines [13]. Even though it is possible to make extensive tables with a computer search, the search of the maximally symmetric subgroup $SU(3)^8$ is simple due to the knowledge of $E_8 \to SU(3)^4$ [1]. To reduce the number of families maximally, we introduce three Wilson lines, i.e. two more Wilson lines in addition to the one

vector	number of states	gauge group
$(\underline{1\ \ -1\ \ 0}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)$	6	$SU(3)_1$
$(0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0)_{I_+}$	1	
$(0\ \ 0\ \ 0\ -1\ -1\ \ 0\ \ 0\ \ 0)_{I_-}$	1	
$(+\ +\ +\ +\ +\ -\ -\ +)_{V_+}$	1	
$(-\ -\ -\ -\ -\ +\ +\ -)_{V_-}$	1	$SU(3)_2$
$(+\ +\ +\ -\ -\ -\ -\ +)_{U_+}$	1	
$(-\ -\ -\ +\ +\ +\ +\ -)_{U_-}$	1	
$(0\ \ 0\ \ 0\ \ 1\ -1\ \ 0\ \ 0\ \ 0)_{I_+}$	1	
$(0\ \ 0\ \ 0\ -1\ \ 1\ \ 0\ \ 0\ \ 0)_{I_-}$	1	
$(+\ +\ +\ +\ -\ +\ +\ -)_{V_+}$	1	
$(-\ -\ -\ -\ +\ -\ -\ +)_{V_-}$	1	$SU(3)_3$
$(+\ +\ +\ -\ +\ +\ +\ -)_{U_+}$	1	
$(-\ -\ -\ +\ -\ -\ -\ +)_{U_-}$	1	
$(0\ \ 0\ \ 0\ \ 0\ \ 0\ \ \underline{1\ -1}\ \ 0)_{I_\pm}$	2	
$(0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ -1\ -1)_{V_+}$	1	
$(0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1)_{V_-}$	1	$SU(3)_4$
$(0\ \ 0\ \ 0\ \ 0\ \ 0\ -1\ \ 0\ -1)_{U_+}$	1	
$(0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 1)_{U_-}$	1	

Table 1: *Root vectors of $SU(3)^4 \subset E_8$. The underlined entries allow permutations. The $+$ and $-$ in the spinor part denote $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. I, V, and U denote the $SU(3)$ spin directions.*

presented in [1],

$$
\begin{aligned}
v &= (0\ \ 0\ \ 0\ \ 0\ \ 0\ \ \tfrac{1}{3}\ \ \tfrac{1}{3}\ \ \tfrac{2}{3})(0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0) \\
a_1 &= (\tfrac{1}{3}\ \ \tfrac{1}{3}\ \ \tfrac{1}{3}\ \ 0\ \ 0\ \ \tfrac{1}{3}\ \ \tfrac{1}{3}\ \ \tfrac{5}{3})(0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0) \\
a_3 &= (0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)(0\ \ 0\ \ 0\ \ 0\ \ 0\ \ \tfrac{1}{3}\ \ \tfrac{1}{3}\ \ \tfrac{2}{3}) \\
a_5 &= (0\ \ 0\ \ 0\ \ 0\ \ 0\ \ \tfrac{2}{3}\ \ \tfrac{2}{3}\ \ \tfrac{4}{3})(\tfrac{1}{3}\ \ \tfrac{1}{3}\ \ \tfrac{1}{3}\ \ 0\ \ 0\ \ \tfrac{1}{3}\ \ \tfrac{1}{3}\ \ \tfrac{5}{3})
\end{aligned}
\tag{2.1}
$$

The unbroken group becomes

$$
SU(3)_1 \times SU(3)_2 \times SU(3)_3 \times SU(3)_4 \times [SU(3)_5 \times SU(3)_6 \times SU(3)_7 \times SU(3)_8]' \tag{2.2}
$$

where the primed $SU(3)$'s have descended from E_8'.

Let us define 27 twisted sectors as following

$$
\begin{gathered}
T0:\ v\ ,\quad T1:\ v + a_1\ ,\quad T2:\ v - a_1, \\
T3:\ v + a_3\ ,\quad T4:\ v - a_3\ ,\quad T5:\ v + a_1 + a_3, \\
T6:\ v + a_1 - a_3\ ,\quad T7:\ v - a_1 + a_3\ ,\quad T8:\ v - a_1 - a_3, \text{etc.}
\end{gathered}
\tag{2.3}
$$

The massless chiral fields obtained from this model are shown in Table 2. The definition of the representation is the same as those given in Ref. [1]. For concreteness, we present the root vectors of Ref. [1] in Table 1. Note that there does not appear massless chiral fields in the untwisted sector.

2.2 Model B

In this subsection, we present another realization of $SU(3)$ trits. Let us introduce following three Wilson lines,

$$\begin{aligned}
v &= (0\ 0\ 0\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{2}{3})(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\
a_1 &= (\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{5}{3})(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\
a_3 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)(0\ 0\ 0\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{2}{3}) \\
a_5 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{5}{3})
\end{aligned} \tag{2.4}$$

The unbroken group becomes

$$SU(3)_1 \times SU(3)_2 \times SU(3)_3 \times SU(3)_4 \times [SU(3)_5 \times SU(3)_6 \times SU(3)_7 \times SU(3)_8]'. \tag{2.5}$$

Similarly, the massless chiral fields are shown in Table 3. In this example, there appear matter fields in the untwisted sector.

3. Construction of supersymmetric standard models

The models presented in the preceding section are just $SU(3)$ trits, and one has to work out more to find out the SSM vacua.

To reduce the number of multiplicities, we used the freedom present in the theory, i.e. the Wilson lines[3]. Introduction of one Wilson line reduces this degeneracy by a factor of 3. The three Wilson line models of the previous section reduced the multiplicity too much, and it is better to remove one Wilson line of the previous $SU(3)$ bit models to obtain three family models. If one removes one Wilson line out of three Wilson lines, the resulting gauge group is certainly enhanced. If it is enhanced, it can be either E_6 or $SU(6) \times SU(2)$ since these have 27 as irreducible representations. The reason why we consider only these two cases is presented in the Dynkin diagram techniques toward orbifold compactifications[13].

By inspecting the Tables, one can easily see which Wilson lines are needed to realize a three family SSM. For this purpose, Model A of the previous section is promising toward trinification. On the other hand, Model B contains the representation $(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$ in the bulk, and is difficult to obtain a trinification spectrum.

Thus, we use Model A for constructing SSM's. In one model(SSM-I) discussed in the following subsection, we easily obtain a three family model. In the other example(SSM-II), we also obtain a three family model. Both models realize an E_6

grand unification with three **27**'s, which can be studied in full detail toward low enersy SUSY phenomenology.

To obtain three families, we must remove one Wilson line so that the degeneracy of fixed points becomes 3. There are two ways to do this, one removing a_1 and the other removing a_3, which are called SSM-I and SSM-II, respectively.

3.1 SSM-I

We choose two Wilson lines a_3 and a_5 from Model A. Thus, our orbifold model is

$$\begin{aligned} v &= (0\ 0\ 0\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{2}{3})(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ a_3 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)(0\ 0\ 0\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{2}{3}) \\ a_5 &= (0\ 0\ 0\ 0\ 0\ \tfrac{2}{3}\ \tfrac{2}{3}\ \tfrac{4}{3})(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{5}{3}) \end{aligned} \tag{3.1}$$

With these shift vectors and Wilson lines, there does not appear matter fields in the untwisted sector. All matter fields arise in the twisted sectors: T0 (v), T1 ($v+a_3$), T2 ($v-a_3$), etc. The massless spectrum conditions in these sectors are the same as those in the corresponding sector of Model A, thus the spectrum from Table 2 can be simply read. This is the reason that Model A contains all the needed code for the matter spectrum. The spectrum is presented in Table 4. In the second column, the $SU(3)$ trits of Table 2 are presented. So, the representation must be written in the enhanced gauge group.

The unbroken gauge group of (3.1) is

$$E_6 \times SU(3)_4 \times [SU(3)_5 \times SU(3)_6 \times SU(3)_7 \times SU(3)_8]'. \tag{3.2}$$

In the third column, the representation content in the enhanced gauge group $E_6 \times SU(3)_4 \times [SU(3)^4]'$ is given. Note that we have an E_6 GUT with three families of **27**. Because E_6 cannot be broken by two independent vacuum expectation values in **27**, we cannot obtain a SSM from the spectrum present in the model. The symmetry breaking pattern and the electroweak hypercharge of this model, SSM-I, will be studied further in the next section, including the Kaluza-Klein(KK) modes.

3.2 SSM-II

Here, we choose a_1 and a_5 as two Wilson lines,

$$\begin{aligned} v &= (0\ 0\ 0\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{2}{3})(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ a_1 &= (\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{5}{3})(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ a_5 &= (0\ 0\ 0\ 0\ 0\ \tfrac{2}{3}\ \tfrac{2}{3}\ \tfrac{4}{3})(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\ 0\ 0\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{5}{3}) \end{aligned} \tag{3.3}$$

With these shift vectors and Wilson lines, there does not appear matter fields in the untwisted sector.

Comparing with SSM-I, we note the striking similarity between these two realizations. If the gauge couplings of E_8 and E_8' are the same, these two models have the interchange symmetry $E_8 \leftrightarrow E_8'$. However, if their gauge couplings are different, SSM-I and SSM-II descibe two different vacua. In any case, SSM-I and SSM-II has the exchange symmetry: SSM-I $\leftrightarrow$ SSM-II, and $g \leftrightarrow g'$. Therefore, we treat only SSM-I.

3.3 SSM-III

There can be another possibility to obtain a trinification spectrum. Out of a few $SU(3)$ factors, we can choose some diagonal $SU(3)$'s by giving VEV's to some link fields. From Table 2, let us try to obtain the following diagonal subgroups,

$$\begin{array}{c} \{SU(3)_1, SU(3)_5\} \\ \{SU(3)_2, SU(3)_4, SU(3)_6\} \\ \{SU(3)_3, SU(3)_8\} \end{array} \tag{3.4}$$

We will interpret $SU(3)_3$ the QCD, $SU(3)_2$ the weak gauge group, and $SU(3)_1$ the remaining factor group $SU(3)_N$ in the trinification unification. Then, by choosing the diagonal subgroups of (3.4), we obtain a trinification in addition to the remaining $SU(3)_7$. If we break the $SU(3)_7$ by VEV's of the T9 trit (1,1,1,1)(1,1,$\bar{3}$,1), then we obtain just the trinification group. Removing vectorlike representations, we obtain the following spectrum under $SU(3)_N \times SU(3)_W \times SU(3)_c$,

$$3\left\{(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}) + (\mathbf{3}, \mathbf{1}, \mathbf{3}) + (\mathbf{1}, \bar{\mathbf{3}}, \bar{\mathbf{3}})\right\}. \tag{3.5}$$

Therefore, we find a vacuum direction where a SSM is realized. But the gauge coupling unification is not naturally implemented, since the three diagonal $SU(3)$ groups do not have the same gauge coupling.

Another problem is that among the identification (3.4) only one relation in the second set is realized by the VEV's of the following link field,

$$(\mathbf{1}, \mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}). \tag{3.6}$$

$$\tag{3.7}$$

In the massless spectrum, we do not have the needed link fields to realize the remaining identifications of (3.4). However, one may use the heavy Kaluza-Klein modes in the bulk for the link fields.

In the remainder of this paper, we concentrate on the SSM-I.

4. Supersymmetric standard model, spontaneous symmetry breaking and electroweak hypercharge

4.1 Hypercharge in $SU(3)_I \times SU(3)_{II} \times SU(3)_{III}$

To ease the discussion, we will name the members of (1.1) in terms of the familiar low energy names. $SU(3)_{III}$ is QCD, the $SU(2)$ subgroup of $SU(3)_{II}$ is the weak $SU(2)$ of the SM, and define the electroweak hypercharge as

$$Y = -\frac{1}{2}(-2T_I + Y_I + Y_{II}) \tag{4.1}$$

where T_I is the third component $(T_3)_I$ of the isospin generators of the group $SU(3)_I$, and Y_K is the $SU(3)_K(K = I, II)$ hypercharge $\frac{2}{\sqrt{3}}(T_8)_I$. The eigenvalues of T and Y are $\{\frac{1}{2}, -\frac{1}{2}, 0\}$ and $\{\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\}$, respectively. The vector indices of $SU(3)_I, SU(3)_{II}$, and $SU(3)_{III}$ are denoted as $M = (1, 2, 3), I = (i, 3)$ and α, respectively. Thus, we identify the three trits of (1.1) in the following way,

$$\begin{aligned}
(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}) = \Psi_l \longrightarrow \Psi_{(\bar{M},I,0)} &= \Psi_{(\bar{1},i,0)}(H_1)_{-\frac{1}{2}} + \Psi_{(\bar{2},i,0)}(H_2)_{+\frac{1}{2}} + \Psi_{(\bar{3},i,0)}(l)_{-\frac{1}{2}} \\
&\quad + \Psi_{(\bar{1},3,0)}(N_5)_0 + \Psi_{(\bar{2},3,0)}(e^+)_{+1} + \Psi_{(\bar{3},3,0)}(N_{10})_0 \qquad (4.2) \\
(\mathbf{1}, \bar{\mathbf{3}}, \mathbf{3}) = \Psi_q \longrightarrow \Psi_{(0,\bar{I},\alpha)} &= \Psi_{(0,\bar{i},\alpha)}(q)_{+\frac{1}{6}} + \Psi_{(0,\bar{3},\alpha)}(D)_{-\frac{1}{3}} \qquad (4.3) \\
(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}}) = \Psi_a \longrightarrow \Psi_{(M,0,\bar{\alpha})} &= \Psi_{(1,0,\bar{\alpha})}(d^c)_{\frac{1}{3}} + \Psi_{(2,0,\bar{\alpha})}(u^c)_{-\frac{2}{3}} + \Psi_{(3,0,\bar{\alpha})}(\overline{D})_{+\frac{1}{3}} \qquad (4.4)
\end{aligned}$$

where N_{10} is the singlet of $SO(10)$ in the $E_6 \to SO(10)$ breaking, and N_5 is the singlet of $SU(5)$ in the $SO(10) \to SU(5)$ breaking. We introduce a name for the above three representations, *humor*. The humor comes in three: *lepton–, quark–, antiquark–humors*. The *humor* is a part of the gauge symmetry in E_6, but in our $SU(3)^3$ it is an independent quantum number.

4.2 E_6 GUT or a trinification

The SSM-I admits two interpretations: one an E_6 grand unification, and the other a trinification plus some extra fields. To see them in terms of a small number of representations, let us break the gauge groups $SU(3)_4$ and $SU(3)'_7$ by VEV's of $(\mathbf{1}, \mathbf{3})$'s and $(\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1})'$'s. Removing vectorlike representations, we obtain the follwing representations transforming as, under the gauge group $E_6 \otimes [SU(3)_I \times SU(3)_{II} \times SU(3)_{III}]'$ where $SU(3)_I \equiv SU(3)^*_5, SU(3)_{II} \equiv SU(3)_6$ and $SU(3)_{III} \equiv SU(3)^*_8$, [4]

$$\begin{aligned}
3\ \{\ &(\mathbf{27}) && (4.5) \\
&\oplus (\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})' \oplus (\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1})' \oplus (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{3})' && (4.6) \\
&\oplus (\mathbf{3}, \mathbf{1}, \mathbf{3})' \oplus (\mathbf{1}, \bar{\mathbf{3}}, \bar{\mathbf{3}})' \oplus 3(\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1})' \oplus 3(\mathbf{1}, \mathbf{3}, \mathbf{1})'\ \} && (4.7)
\end{aligned}$$

[4] The complex conjugate symbol * is that the anti-fundamental $\bar{\mathbf{3}}$ of $SU(3)'_5$ is interpreted as the fundamental representation $\mathbf{3}$ of $SU(3)_I$, etc.

If we interpret the E_8 part as the observable sector, we obtain an E_6 grand unification as given in (4.5). If we interpret the E_8' part as the observable sector, then we obtain the trinification spectrum in (4.6) and some extra fields of (4.7).

To clarify whether the above trinification is an allowable one, let us check the $\sin^2\theta_W^0$ for the observable E_8' case. The trinification spectrum (4.6) is the same as the one given in (1.1), and hence the hypercharge given in Eq. (4.1) gives the SM hypercharges from the above trinification spectrum. Now let us observe what are the hypercharges of the extra fields of Eq. (4.7). The $SU(2)\times U(1)_Y\times SU(3)_c$ representation contents of one extra family of (4.7) are

$$\begin{aligned}
(\mathbf{3},\mathbf{1},\mathbf{3})' &= (1,3)_{1/3}+(1,3)_{-2/3}+(1,3)_{1/3}\\
(\mathbf{1},\overline{\mathbf{3}},\overline{\mathbf{3}})' &= (2,\bar{3})_{1/6}+(1,\bar{3})_{-1/3}\\
3(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})' &= 3(1,1)_{1/3}+3(1,1)_{-2/3}+3(1,1)_{1/3}\\
3(\mathbf{1},\mathbf{3},\mathbf{1})' &= 3(2,1)_{1/6}+3(1,1)_{-1/3}
\end{aligned}\tag{4.8}$$

Thus, the contribution to the numerator and the denominator of $\mathrm{Tr}\ T_3^2/\mathrm{Tr}\ Q_{em}^2$ is

$$\frac{3(\frac14+\frac14+\frac14+\frac14)}{3(\frac19+\frac49+\frac19+\frac49+\frac19+\frac19+\frac19+\frac49+\frac19+\frac49+\frac19+\frac19)}=\frac{3}{8},$$

whence the GUT value of $\sin^2\theta_W^0$ is not changed from $\frac38$, and we do can obtain a coupling unification [1], even though the extra fields are present. Note, however, that there survive weirdly charged leptons down to low energy. The extra fields have three more families of quarks which do not mix with the trinification spectrum. This model is a kind of two village model, envisioned in Ref. [1]. The QCD coupling constant is not asymptotically free above the electroweak symmetry breaking scale, and hence this model has another problem of coupling constant unification at 2×10^{16} GeV. However, unification at an intermediate scale is a possibility.

4.3 E_6 GUT and spontaneous symmetry breaking

The model presented as SSM-I with the observable E_8 in Sec. 3 is an E_6 model with three **27**'s. This section is mostly devoted to the group theory nature of the exceptional E_6 and E_8 groups.

Let us first discuss the spontaneous symmetry breaking.

We need extra fields, $\mathbf{27}_h+\overline{\mathbf{27}}_h$, which develop VEV's for a doublet-triplet splitting mechanism. For the gauge symmetry breaking of E_6, we need an adjoint representation. The necessity of the adjoint representation in $E_6, SO(10)$, and $SU(5)$ toward SM is the well-known fact. The reason is the following.

Suppose that three **27**'s acquire VEV's. A VEV of **27** lowers the rank–6 E_6 to rank–5 groups. For one **27**, we can always choose the vacuum direction so that an $SO(10)$ is unbroken. Under the unbroken subgroup, **27** branches into

$$\mathbf{27}\longrightarrow\mathbf{1}+\mathbf{10}+\mathbf{16}.\tag{4.9}$$

The adjoint $\mathbf{78}$ of E_6 branches into

$$\mathbf{78} \longrightarrow \mathbf{45} + \mathbf{16} + \overline{\mathbf{16}} + \mathbf{1}. \tag{4.10}$$

Then we observe that VEV's of $\mathbf{27}$'s cannot make all the $E_6/SO(10)$ coset space gauge bosons(the vectorlike $\mathbf{16} + \overline{\mathbf{16}}$) heavy. This is the reason that we must introduce a vectorlike representation $\mathbf{27}_h + \overline{\mathbf{27}}_h$ which develop VEV's. Introduction of $\mathbf{27}_h$ and $\overline{\mathbf{27}}_h$ is allowed in our Z_3 orbifold compactification. In obtaining the massless spectrum, we used the GSO-like projection and listed only massless fields in Table 2. However, the projected out fields are actually the massive modes and these are the heavy Kaluza-Klein(KK) modes such as $\mathbf{27}_h + \overline{\mathbf{27}}_h$. Simply, they cannot remain massless. Thus, we can introduce them with a large mass parameter such as $M_{KK}\mathbf{27}_h \cdot \overline{\mathbf{27}}_h$. Then, we can write a supersymmetric term of the form

$$M_{KK}\mathbf{27}_h\overline{\mathbf{27}}_h + \overline{\mathbf{27}}_h \cdot \overline{\mathbf{27}}_h \cdot \overline{\mathbf{27}}_h \tag{4.11}$$

so that $\Psi_{(\bar{3},3,0)}$ and $\Psi_{(3,\bar{3},0)}$ member of $\mathbf{27}_h$ and $\overline{\mathbf{27}}_h$ develop VEV's of order M_{KK}. Then, we have some needed vectorlike Goldstone modes to make the $E_6/SO(10)$ coset gauge bosons heavy.

After assigning VEV's in the $\langle\Psi_{h(\bar{3},3,0)}\rangle$ and $\langle\bar{\Psi}_{h(3,\bar{3},0)}\rangle$ directions of $(\mathbf{27}_h + \overline{\mathbf{27}}_h)$, the other $\mathbf{27}$'s lose a lot of gauge degrees of freedom to change directions. Under this circumstance, suppose that we can relocate the fields such that even a flipped $SU(5)$ assignment [16] is realized. The flipped $SU(5)$ in our trits terminology is to gather $\Psi_{(\bar{3},i,0)}$ and $\Psi_{(2,0,\bar{\alpha})}$ in $\overline{\mathbf{5}}$ of $SU(5)$, and $\Psi_{(0,\bar{i},\alpha)}, \Psi_{(\bar{1},3,0)}$ and $\Psi_{(1,0,\bar{\alpha})}$ in $\mathbf{10}$ of $SU(5)$, and $\Psi_{(\bar{2},3,0)}$ in the singlet of $SU(5)$. By giving a VEV to $\langle\Psi_{(\bar{1},3,0)}\rangle$ which belongs to $\mathbf{10}$ of $SU(5)$, we can break down to the standard model gauge group. If successful, this scenario would not need an adjoint representation. However, it does not work because of the wrong hypercharge as shown below.

Of course, with one pair $(\mathbf{27}_h + \overline{\mathbf{27}}_h)$ the gauge group breaks down to $SO(10)$ only, not to $SU(5) \times U(1)$. The above relocation amounts to introducing an adjoint representation of $SO(10)$ since the number of gauge degrees of freedom is reduced from 45 to 25. Namely, 20 Goldstone bosons are added in this relocation. One may be tempted to interpret $\mathbf{10}$ plus $\overline{\mathbf{10}}$ of (4.9) as the needed 20 Goldstone modes. However, the hypercharges do not match nicely.

The problem is the following. The frequently cited chain $E_6 \to SO(10) \to SU(5)$ contains the so-called colored X and Y gauge bosons of $SU(5)$, with the electromagnetic charges $\frac{4}{3}$ and $\frac{1}{3}$, respectively. These form a colored doublet with $Y = \frac{5}{6}$. In particular, the relocation amounts to introducing colored Goldstone bosons with charge $\pm\frac{4}{3}$ which is not contained in the representation (1.1). Thus, we cannot supply all the needed Goldstone modes for the relocation with $(\mathbf{27}_h + \overline{\mathbf{27}}_h)$. Note that there is another kind of $\mathbf{27}$ represented in an anomaly free trits combination as

$$\mathbf{27}' \equiv (\overline{\mathbf{3}}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \overline{\mathbf{3}}, \overline{\mathbf{3}}) + (\mathbf{3}, \mathbf{1}, \mathbf{3}). \tag{4.12}$$

Again, $\mathbf{27}'$ does not contain a colored $Q_{em} = \pm\frac{4}{3}$ component. Thus, $\mathbf{27}_h$ and $\overline{\mathbf{27}}_h$ cannot break E_6 down to the SM gauge group.

Since the bulk fields originated from the adjoint representation of E_8, in the bulk there must be KK mode scalars with the E_6 adjoint quantum numbers. By orbifolding with three Wilson lines, these are all projected out, which means that they are heavy. We have started from a three Wilson line model Table 2 where there is no massless $Q_{em} = \pm\frac{4}{3}$ scalar. But, the E_6 group is broken there with three Wilson lines, which means that $Q_{em} = \pm\frac{4}{3}$ gauge bosons became heavy. In terms of the Higgs mechanism, we can view Table 2 as containing massless $X, \overline{X}$ gauge bosons and their longitudinal massless colored scalars(the Goldstone modes) $x, \bar{x}$ with $Q_{em} = \pm\frac{4}{3}$. Now, by removing one Wilson line and going into a two Wilson line model(Table 4), we observed that the $SU(3)^3$ gauge symmetry is enhanced to E_6. This means that the initially heavy $X, \overline{X}$ gauge bosons become massless. In terms of the Higgs mechanism, the Goldstone mode $x, \bar{x}$ must become heavy to be decoupled from the massless gauge bosons $X, \overline{X}$. Thus, in the bulk spectrum with two Wilson lines there must be heavy $x, \bar{x}$ with $Q_{em} = \pm\frac{4}{3}$. These hidden $Q_{em} = \pm\frac{4}{3}$ particles with two Wilson lines are not listed in the orbifold tables with two Wilson lines which do not include the heavy KK modes of an adjoint scalar.

As a low energy effective theory, we can consider two possibilities. One is that the gauge symmetry is not enhanced to E_6. Simply we have not counted the massless spectrum in the bulk, for example $x, \bar{x}$. If we count them, we have an $SU(3)^3$ theory. But the string calculation with two Wilson lines excludes this possibility. The second possibility is that the gauge symmetry is in fact enhanced. But we have to consider the heavy bulk chiral fields with the adjoint quantum numbers, i.e. $\Sigma \equiv \mathbf{78}$, as commented in the preceding paragraph. Since Σ is a KK mode with mass M, we can consider a superpotential $M\mathrm{Tr}\Sigma^2$. If one can introduce a cubic superpotential of Σ such as $\mathrm{Tr}\Sigma^3$, this heavy adjoint field develops a VEV and chooses the vacuum diection to $SU(3)^3$ which was shown to be not broken even with three Wilson lines. Therefore, it is appropriate to consider $SU(3)^3$ at low energy. The importance of $\mathbf{78}$ is allowing a direction to $SU(5)\times U(1)$ [16], instead of $SO(10)$. Namely, our relocation of the fields is allowed with $\mathbf{78}$. In this case of using Σ, we do not use the Goldstone bosons arising from $\langle(\mathbf{27}+\overline{\mathbf{27}})_h\rangle$ for braking E_6 down to $SU(3)^3$. But the spectrum $(\mathbf{27}+\overline{\mathbf{27}})_h$ or $(\mathbf{27}+\overline{\mathbf{27}})'_h$ is needed for the breaking of $SU(3)^3$ down to the SM. Therefore, we will consider them for further gauge symmetry breaking and the doublet-triplet splitting.

Note that it is frequently said that it is difficult to obtain massless adjoint representations in the orbifold compactification. However, the adjoint chiral field with heavy KK towers is a possibility, and we speculated that they can break the gauge symmetry. Previously, only a flat direction of massless scalars has been searched. It was possible for us to guess this kind of phenomenon because we obtained the most asymmetric vacuum with three Wilson lines first and then studied the two Wilson

line model with an enhanced symmetry in an effective theory framework.

5. Trits algebra

So far we considered the trits $\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}$ of $SU(3)$ groups. It turns out that the trits seems to be useful for studying the low lying representations of exceptional groups. Therefore, this section is devoted to the trits algebra.

For E_7, we have to introduce $SU(2)$ factor and can properly generalize the trits just for one factor group, $SU(2)_4$ instead of $SU(3)_4$. Trits do not include higher representations of $SU(3)$, e.g. **6, 10, 15,** etc.

For humor zero representations of E_6, we include $(8,1,1),(1,8,1),(1,1,8)$, and $(1,1,1)$ only. Then this trits system closes under multiplication.

5.1 Hypercharges of the trits of the E_6 adjoint representation

From the trits we have introduced so far, we cannot see $Q_{em} = \pm\frac{4}{3}$ particles. In fact, the $Q_{em} = \pm\frac{4}{3}$ particles arise in the adjoint representation. The adjoint representation of E_6 is picked up from 729 entries of $\overline{\mathbf{27}}\times\mathbf{27}$. Here, we represent them in terms of trits so that E_6 can be studied in most aspects in terms of trits and the familiarity of $SU(3)$ can be useful for future studies of exceptional groups. The trits multiplication of $\overline{\mathbf{27}}\times\mathbf{27}$ gives

$$\begin{aligned}&\{(\bar{6}+3,3,3)+(3,\bar{6}+3,3)+(3,3,\bar{6}+3)+\text{complex conjugate}\}\\ &\quad+(8+1,8+1,1)+(1,8+1,8+1)+(8+1,1,8+1).\end{aligned}$$

It is obvious what should be picked up from 1 and 8, i. e. $(8,1,1)+(1,8,1)+(1,1,8)$. Since we do not have any higher representations, we pick up 3 from $(\bar{6}+3)$. But, then the number count for the adjoint shows that there is a factor 3 too much in the first line. Here, we want to streamline the notation. $(\bar{6}+3,3,3)$ is in fact $((\bar{3}\times\bar{3})_s+\bar{3}\wedge\bar{3},3,3)$. So the 3 in the first entry of $(\bar{6}+3,3,3)$ is antisymmetric combination of two $\bar{3}$'s, $\bar{3}\wedge\bar{3}$. This $\bar{3}\wedge\bar{3}$ is designed to kill $\bar{3}$ by taking a skew product, which means that we always take an antisymmetric combination.(This antisymmetric multiplication applies to the tensor representation for adjoint also.) Thus, when we write $(\bar{3}\wedge\bar{3},3,3)$, we should interpret it as having 9 elements. The first entry $\bar{3}\wedge\bar{3}$ kills $\bar{3}$. Then, one of the remaining two 3's must convert 3 to $\bar{3}$ so that **27** is obtained by operating $(\bar{3}\wedge\bar{3},3,3)$ on **27**. This operation must have nine elements. For example, let us consider $(\bar{3},3,1)$ element of **27**. It is changed to, according to the above rule, $(1,\bar{3},3)$ which belongs to **27**. Here, it is obvious that the transition of $3\to\bar{3}$ must be counted as one, not three. In our notation, when the first $\bar{3}\wedge\bar{3}$ kills the $\bar{3}$ in the first entry, the second entry 3 must be converted to $\bar{3}$. The third entry is 1, and it is changed to 3. Thus, the changes in the first entry and the third entry have multiplicity 3 each. Then the change in the second entry must have multiplicity

one. Indeed, it can be interpreted in this way if we view the change $3 \to \bar{3}$ as an inversion. This inversion is automatically included if the multiplication in the second entry is also an antisymmetric choice. Namely, this antisymmetric choice has multiplicity one. Thus, understanding every group multiplication is antisymmetric, we can represent the above operation as $(\bar{3} \wedge \bar{3}, I(3), 3)$, which symbolically depicts nine elements. Alternatively, we can represent it as $(I, 3, 3)$ which also shows nine elements, but the location of the actual degrees of freedom is hided. The advantage of this latter notation is that there is no side remark on inversion as in the former case, and just the *antisymmetric multiplication or the wedge product* is all we need for the manipulation. Therefore, we use the latter notation below. With the notation $\mathbf{I}$, we represent the highest(absolute value) hypercharge of the triplet as a subscript and the representation content such as $\bar{3} \times \bar{3} = 3$ in the bracket. Note that the entry belong to $\mathbf{I}$ is going to be killed, which is emphasized by a bold character. From now on trits multiplication is always understood as a wedge product. Thus, we obtain the following adjoint representation, including the *charged* trits,

$$\begin{aligned} \mathbf{78} = &\ (\mathbf{8}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{8}) \\ &+ (\mathbf{I}_{-\frac{2}{3}}(\mathbf{3}), \mathbf{3}, \mathbf{3}) + (\mathbf{3}, \mathbf{I}_{+\frac{1}{3}}(\mathbf{3}), \mathbf{3}) + (\mathbf{3}, \mathbf{3}, \mathbf{I}_{\mathbf{0}}(\mathbf{3})) \\ &+ (\mathbf{I}_{+\frac{2}{3}}(\bar{\mathbf{3}}), \bar{\mathbf{3}}, \bar{\mathbf{3}}) + (\bar{\mathbf{3}}, \mathbf{I}_{-\frac{1}{3}}(\bar{\mathbf{3}}), \bar{\mathbf{3}}) + (\bar{\mathbf{3}}, \bar{\mathbf{3}}, \mathbf{I}_{\mathbf{0}}(\bar{\mathbf{3}})) \end{aligned} \tag{5.1}$$

where we have explicitly indicated the hypercharge in the subscripts of $\mathbf{I}$. $\mathbf{I}$ implies one multiplicity, kills $\mathbf{3}$ or $\bar{\mathbf{3}}$ in an $SU(3)$ it is located, but creates multiplicity three at another $SU(3)$ by *inverting* $\mathbf{3}$ or $\bar{\mathbf{3}}$. We have also shown the representation content in the bracket from which representation we picked it up. The representations containing $\mathbf{I}$ are the humor changing ones. We observe that the colored $Y = \pm\frac{5}{6}$ doublets appear in $(\mathbf{I}_{+\frac{2}{3}}, \bar{\mathbf{3}}, \bar{\mathbf{3}}) + (\mathbf{I}_{-\frac{2}{3}}, \mathbf{3}, \mathbf{3})$ which contains X and $\overline{X}$. The removed components from 729 form the representation $\mathbf{1}+\mathbf{650}$ of E_6.

In Eq. (5.1), we could have used the $SU(3)_3$ hypercharge $Y_3 = \text{diag.}(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ to show explicitly which combination was meant in $\mathbf{I}_{\mathbf{0}}(\mathbf{3})$ and $\mathbf{I}_{\mathbf{0}}(\bar{\mathbf{3}})$. Then, they would have been $(\mathbf{3}, \mathbf{3}, \mathbf{I}_{-\frac{2}{3}}(\mathbf{3}))$ and $(\bar{\mathbf{3}}, \bar{\mathbf{3}}, \mathbf{I}_{+\frac{2}{3}}(\bar{\mathbf{3}}))$. Interpreting the electroweak hypercharge as given in Eq. (4.1), we obtain the usual unbroken QCD. Interpreting the electroweak hypercharge as $Y_{HN} = Y + Y_3$, we obtain the Han-Nambu quarks.

Now let us proceed to show the group multiplication of E_6. Usual multiplication of a singlet is $\mathbf{1} \times \mathbf{3} = \mathbf{3}$ at the same $SU(3)$. But, for the multiplication in E_6 with $\mathbf{I}$, $\mathbf{I}(\mathbf{3}) \times \mathbf{3}$ is $\bar{\mathbf{3}}$ but at a different location of $SU(3)$. In this way, the adjoint changes $\mathbf{27}$ to $\mathbf{27}$, which can be checked explicitly. This is not the usual group multiplication in $SU(N)$ groups. It is a specific choice in the exceptional groups. Since the inverting operator $\mathbf{I}$ carry the subscripts(the hypercharge), it picks up only the hypercharge matching transitions. As an example, let us find out, by $(\mathbf{I}_{-\frac{2}{3}}(\mathbf{3}), \mathbf{3}, \mathbf{3})$ in $\mathbf{78}$, what

will be the allowed transition of the following **27**:

$$\begin{aligned}
(\bar{\mathbf{3}},\mathbf{3},\mathbf{1})=\Psi_{(\bar{M},I,0)}&=\Psi_{(\bar{1},i,0)}(H_1)_{-\frac{1}{2}}+\Psi_{(\bar{2},i,0)}(H_2)_{+\frac{1}{2}}+\Psi_{(\bar{3},i,0)}(l)_{-\frac{1}{2}}\\
&\quad+\Psi_{(\bar{1},3,0)}(N_5)_0+\Psi_{(\bar{2},3,0)}(e^+)_{+1}+\Psi_{(\bar{3},3,0)}(N_{10})_0\\
(\mathbf{1},\bar{\mathbf{3}},\mathbf{3})=\Psi_{(0,\bar{I},\alpha)}&=\Psi_{(0,\bar{i},\alpha)}(q)_{+\frac{1}{6}}+\Psi_{(0,\bar{3},\alpha)}(D)_{-\frac{1}{3}}\\
(\mathbf{3},\mathbf{1},\bar{\mathbf{3}})=\Psi_{(M,0,\bar{\alpha})}&=\Psi_{(1,0,\bar{\alpha})}(d^c)_{\frac{1}{3}}+\Psi_{(2,0,\bar{\alpha})}(u^c)_{-\frac{2}{3}}+\Psi_{(3,0,\bar{\alpha})}(\overline{D})_{+\frac{1}{3}}.
\end{aligned}$$

We obtain $(\bar{\mathbf{3}}\times\bar{\mathbf{3}}\times\bar{\mathbf{3}},\mathbf{3}\times\mathbf{3},\mathbf{3})$,[5] $(\mathbf{I}_{-\frac{2}{3}}(\mathbf{3}),\mathbf{3}\times\bar{\mathbf{3}},\mathbf{3}\times\mathbf{3})$, and $(\mathbf{I}_{-\frac{2}{3}}(\mathbf{3})\times\mathbf{3},\mathbf{3},\mathbf{3}\times\bar{\mathbf{3}})$ by group multiplication. Among these the last two do not belong to **27** and we exclude them in that it is not the allowed direction.[6] The first one is $(\mathbf{1},\bar{\mathbf{3}},\mathbf{3})$ which is a member of **27** listed above. Thus, the member $(\mathbf{I}_{-\frac{2}{3}}(\mathbf{3}),\mathbf{3},\mathbf{3})$ in **78** transforms the member $(\bar{\mathbf{3}},\mathbf{3},\mathbf{1})$ in **27** into the member $(\mathbf{1},\bar{\mathbf{3}},\mathbf{3})$ in **27**. In this transition the $\overline{X}$ gauge boson transforms H_2^+ to $D(-\frac{1}{3})$, for example. In the full E_6 group, we have to consider this kind of *humor* transitions. [7] But in our $SU(3)^3$ theory, we need only $(\mathbf{8},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{8},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{8})$ for the member of the adjoint representation.

Since we have shown explicitly that the **I** operators change humor, we can now discuss what the hypercharge shown in the subscript means. For $\mathbf{I}_{-\frac{2}{3}}(\mathbf{3})$ the set of hypercharges is $\{-\frac{2}{3},\frac{1}{3},\frac{1}{3}\}$ since we have written the largest magnitude. When we kill $\bar{\mathbf{3}}$ from $SU(3)_1$, it kills the hypercharges $\{\frac{2}{3},-\frac{1}{3},-\frac{1}{3}\}$ of $\bar{\mathbf{3}}$ of $SU(3)_1$ and creates $\{\frac{2}{3},-\frac{1}{3},-\frac{1}{3}\}$ at some other $SU(3)$. Therefore, the hypercharges added for the creation process must be shown. For the humor transition $(\bar{\mathbf{3}},\mathbf{3},\mathbf{1})\to(\mathbf{1},\bar{\mathbf{3}},\mathbf{3})$, there are nine hypercharge changing cases, and we must consider all the cases with $\{\frac{2}{3},-\frac{1}{3},-\frac{1}{3}\}$. Thus, the $SU(2)_W$–doublet, color–triplet, and humor–changing transitions are possible with hypercharges $\frac{5}{6},-\frac{1}{6},-\frac{1}{6}$, among which $\frac{5}{6}$ corresponds to the X,Y gauge boson doublet of $SU(5)$. In E_6, there are two more colored doublets implied by $\{-\frac{1}{6},-\frac{1}{6}\}$, but in the $SU(5)$ subgroup they do not appear. This raises a question on the number of generators. We can see that the number counting of $(\mathbf{I},\mathbf{3},\mathbf{3})$ is nine in this form. But as explained above, the entry **I** has three components, and it looks like we have 27 members in **(I,3,3)**. But, looking at the operation, one of **3** is conveted to $\bar{\mathbf{3}}$, which is just an inversion and counts as one. Therefore, the operation $(\mathbf{I},\mathbf{3},\mathbf{3})$ has 9 elements.

As another example, consider the tensor product **27**×**27**. In terms of trits, we separate the symmetric and antisymmetric combinations first, and obtain

$$\mathbf{27}\times\mathbf{27}=\left[\frac{27(27+1)}{2}\right]_s+\left[\frac{27(27-1)}{2}\right]_a=\overline{\mathbf{27}}_s+\mathbf{351}_s+\mathbf{351}_a, \tag{5.2}$$

[5] We note that $Y=0$ is picked up from $\mathbf{I}_{-\frac{2}{3}}(\mathbf{3})\times\bar{3}$ since it is basically the singlet selection from $\bar{\mathbf{3}}\times\bar{\mathbf{3}}\times\bar{\mathbf{3}}$.

[6] In $SU(N)$ groups, all members of the fundamental representation are connected by some untary transformation. In exceptional groups, certainly it is not so.

[7] For low lying representations(the fundamental and adjoint representations), our trits treatment is much simpler than the more complete studies[17].

where in $\overline{\mathbf{27}}$ we obtain the exchange symmetry from two antisymmetric factors,

$$\begin{aligned}\overline{\mathbf{27}}_s &= [(\bar{3},3,1)\cdot(\bar{3},3,1)+(1,\bar{3},3)\cdot(1,\bar{3},3)+(3,1,\bar{3})\cdot(3,1,\bar{3})]_{s\ from\ a's}\\ &= (3_a,\bar{3}_a,1)+(1,3_a,\bar{3}_a)+(\bar{3}_a,1,3_a).\end{aligned}\tag{5.3}$$

5.2 Trits representation of E_8 adjoint

Since we observed that the trits are extremely useful in manipulating the exceptional group algebra, in this subsection we list the trits of the adjoint representation of E_8. For this, it is important to note a maximal subgroup $E_6\times SU(3)$ of E_8. It can be seen easily from the Dynkin diagram technique[13]. The extended E_8 Dynkin diagram is shown in Fig. 1.

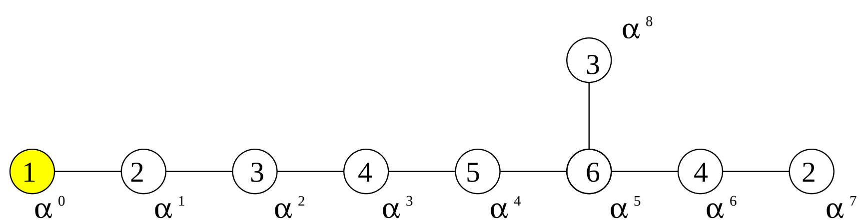

Figure 1: The extended Dynkin diagram of E_8 group. The numbers in the circle are the Coxeter labels n_i of the corresponding simple roots.

Here, α's represent simple roots. From this extended Dynkin diagram, we obtain $E_6\times SU(3)$ by removing the simple root α_2. Then, we can see where each $SU(3)$ factors of ours came from. Our $SU(3)_4$ is generated by α_0 and α_1. The E_6 is generated by α_i with $(i=3,4,\cdots,8)$. The subgroup $SU(3)^3$ of E_6 is obtained from the extended E_6 Dynkin diagram in which α_9 is attached to the α_8 of the E_6 subgroup. From this extended E_6 Dynkin diagram, remove α_5 to obtain $SU(3)^3$ which is generated by three sets $\{\alpha_3,\alpha_4\},\{\alpha_6,\alpha_7\}$ and $\{\alpha_8,\alpha_9\}$. Thus, there exists an interchange symmetry of three $SU(3)$ factors, namely among $SU(3)_1, SU(3)_2$ and $SU(3)_3$.

We know that the adjoint representation **78** of E_6 and the adjoint representation **8** of $SU(3)_4$ must belong to **248**. The **78** is given in Eq. (5.1) and **8** of $SU(3)_4$ is given in Table 1. The remaining components of **248** are $162=81\times 2$. In string theory, the removed components from **248** by orbifolding must be the ones in the bulk. Indeed, in Model B and in Ref. [1] we observed such a bulk field. It is $3(\bar{3},3,1,\bar{3})$ which has 81 components. However, if we have not orbifolded, these three identical ones must have respected the interchange symmetry of the three $SU(3)$ factors in E_6. Taking into account the fact that **248** is real, we must supply the complex conjugated fields

also. Therefore, the components of **248** are

$$\begin{aligned}\mathbf{248} =\ & (8,1,1;1)+(1,8,1;1)+(1,1,8;1)+(1,1,1;8)\\ &+(\mathbf{I}_{-\frac{2}{3}}(\mathbf{3}),\mathbf{3},\mathbf{3};\mathbf{1})+(\mathbf{3},\mathbf{I}_{+\frac{1}{3}}(\mathbf{3}),\mathbf{3};\mathbf{1})+(\mathbf{3},\mathbf{3},\mathbf{I_0}(\mathbf{3});\mathbf{1})\\ &+(\mathbf{I}_{+\frac{2}{3}}(\bar{\mathbf{3}}),\bar{\mathbf{3}},\bar{\mathbf{3}};\mathbf{1})+(\bar{\mathbf{3}},\mathbf{I}_{-\frac{1}{3}}(\bar{\mathbf{3}}),\bar{\mathbf{3}};\mathbf{1})+(\bar{\mathbf{3}},\bar{\mathbf{3}},\mathbf{I_0}(\bar{\mathbf{3}});\mathbf{1})\\ &+(\bar{3},3,1;\bar{3})+(1,\bar{3},3;\bar{3})+(3,1,\bar{3};\bar{3})\\ &+(3,\bar{3},1;3)+(1,3,\bar{3};3)+(\bar{3},1,3;3).\end{aligned} \tag{5.4}$$

In Eq. (5.4), the highlighted trits show the *exceptional group* nature, [8] for which a special care must be taken into account in the group multiplication.

5.3 Trits representations of $SU(5)$ and $SO(10)$ subgroups of exceptional groups

For the subgroups of the exceptional groups, we can choose the trit elements of E_6 representations such that a fundamental representation of the subgroup is formed. For the adjoint representation, we must choose the relevant ones from the highlighted elements in (5.4) plus the usual ones from the octet pieces. We show this for the $SU(5)$ and $SO(10)$ subgroups of E_6.

For **5** of $SU(5)$, we choose the following from **27**,

$$\begin{pmatrix}\Psi_{(0,\bar{3},\alpha)}(D)_{-\frac{1}{3}}\\ \Psi_{(\bar{2},i,0)}(H_2)_{+\frac{1}{2}}\end{pmatrix} \tag{5.5}$$

For the adjoint representation, referring to (5.1), we choose the following

$$\begin{pmatrix}(1,1,8) & (\mathbf{I}_{-\frac{2}{3}}(3),2_{-\frac{1}{6}},3)\\ (\mathbf{I}_{+\frac{2}{3}}(\bar{3}),\bar{2}_{\frac{1}{6}},\bar{3}) & (1,3,1)\end{pmatrix} \tag{5.6}$$

plus the singlet hypercharge

$$Y=\begin{pmatrix}-\frac{1}{3}I_{3\times3} & 0\\ 0 & +\frac{1}{2}I_{2\times2}\end{pmatrix} \tag{5.7}$$

which has to be normalized by multiplying $\sqrt{\frac{3}{5}}$.

For **10** of $SO(10)$, we choose the following from **27**,

$$\begin{pmatrix}\Psi_{(0,\bar{3},\alpha)}(D)_{-\frac{1}{3}}\\ \Psi_{(\bar{2},i,0)}(H_2)_{+\frac{1}{2}}\\ \Psi_{(3,0,\bar{\alpha})}(\overline{D})_{+\frac{1}{3}}\\ \Psi_{(\bar{1},i,0)}(H_1)_{-\frac{1}{2}}\end{pmatrix} \tag{5.8}$$

[8] Exceptional groups are used at the field theory level for grand unification[18]. In E_7, the chirality issue was not treated.

For the adjoint representation, referring to (5.1), we choose the following

$$\begin{pmatrix} (1,1,8) & (\mathbf{I}_{-\frac{2}{3}}(3), 2_{-\frac{1}{6}}, 3) & (\bar{1}_{-\frac{1}{3}}, \mathbf{I}_{-\frac{1}{3}}(\bar{3}), \bar{3}) & (3, 2_{\frac{1}{6}}, \mathbf{I}_0(3)) \\ (\mathbf{I}_{+\frac{2}{3}}(\bar{3}), \bar{2}_{\frac{1}{6}}, \bar{3}) & (1,3,1) & (3, 2_{\frac{1}{6}}, \mathbf{I}_0(3)) & (I^+(8)_{+1}, 1, 1) \\ (1_{\frac{1}{3}}, \mathbf{I}_{\frac{1}{3}}(3), 3) & (\bar{3}, \bar{2}_{-\frac{1}{6}}, \mathbf{I}_0(\bar{3})) & (1,1,8) & (\mathbf{I}_{\frac{2}{3}}(\bar{3}), \bar{2}_{\frac{1}{6}}, \bar{3}) \\ (\bar{3}, \bar{2}_{-\frac{1}{6}}, \mathbf{I}_0(\bar{3})) & (I^-(8)_{-1}, 1, 1) & (\mathbf{I}_{-\frac{2}{3}}(3), 2_{-\frac{1}{6}}, 3) & (1,3,1) \end{pmatrix} \tag{5.9}$$

where $I^{\pm}(8)$ are the members of the I spin raising and lowering operators in the octet.[9] The multiplicity of the representation is denoted by 3, 2, 1, $\bar{3}, \bar{2}, \bar{1}$. These also show the representations $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $SU(3)$ from which they came from. If $\mathbf{3}$ and $\bar{\mathbf{3}}$ are split into 2 and 1, we showed the hypercharges of the corresponding representation by subscripts. $\mathbf{I}$ *counts one multiplicity, but it changes the humor.* We have to add two more diagonal generators to make up 45 members of the $SO(10)$ adjoint. One is the hypercharge

$$Y = \begin{pmatrix} -\frac{1}{3}I_{3\times 3} & 0 & 0 & 0 \\ 0 & +\frac{1}{2}I_{2\times 2} & 0 & 0 \\ 0 & 0 & +\frac{1}{3}I_{3\times 3} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}I_{2\times 2} \end{pmatrix} \tag{5.10}$$

and the other is

$$Y_{B-L} = \begin{pmatrix} +\frac{1}{3}I_{3\times 3} & 0 & 0 & 0 \\ 0 & -I_{2\times 2} & 0 & 0 \\ 0 & 0 & -\frac{1}{3}I_{3\times 3} & 0 \\ 0 & 0 & 0 & +I_{2\times 2} \end{pmatrix}. \tag{5.11}$$

6. Yukawa couplings and doublet-triplet splitting

The massless field $\mathbf{27}$ of Table 4 can have the following Yukawa couplings,

$$-\mathcal{L}_Y = \frac{1}{3!} f_{abc} \Psi^a \Psi^b \Psi^c \tag{6.1}$$

where a, b, c contain family indices. Note that f_{abc} is *completely symmetric*. In our scheme we introduced 3 families and one heavy $\mathbf{27}_h$ which also participate in the coupling. Since we want to assign large VEV's only to $(\mathbf{27}_h + \overline{\mathbf{27}}_h)$, for the doublet triplit splitting, we consider

$$-\mathcal{L}_h = \frac{1}{3!} f_{ab} \Psi^a \Psi^b \Psi_h \tag{6.2}$$

where Ψ_h is $\mathbf{27}_h$.

Since Ψ^a's appear in T0, we can consider that three families are identical as far as Z_3 orbifolding is concerned, i.e. they obtain the same phase under the Z_3 shift.

[9] This I spin notation should not be confused with the humor changing operator $\mathbf{I}$.

But in the internal space they are actually located at three different fixed points, which may lead to nontrivial texture for fermion masses. Inserting VEV's in the direction

$$\langle \Psi^h_{(\bar{1},3,0)} \rangle = V, \tag{6.3}$$

many components in $3\cdot(\mathbf{27})$ are removed.

Before showing the doublet-triplet splitting explicitly, we point out that the resolution of this doublet-triplet splitting problem in the flipped $SU(5)$ model heavily assumes the absence of H_1H_2 coupling. It is the familiar μ problem, and can be solved by introducing a Peccei-Quinn symmetry[5]. But in string theory, we can see that the H_1H_2 term cannot arise at the tree level. Since both H_1 and H_2 belong to $\mathbf{27}$ in our compactification, a guessed term for H_1H_2, i.e. a term among light fields $\mathbf{27}\cdot\mathbf{27}$ is not allowed. However, $\mathbf{27}\cdot\mathbf{27}\cdot\mathbf{27}$ is allowed and H_1H_2 must be forbidden from this cubic term. Thus, a string resolution of the μ problem is as simple as this [19] under the assumption that there exists a mechanism for the doublet-triplet splitting.

The VEV given in (6.3) allow the following two types of nonvanishing terms. One is coming from considering $SU(3)^3$ singlet by taking three different trits from Ψ^a, Ψ^b, and Ψ_h. In this case, D and $\overline{D}$ of $\mathbf{27}$ are removed at the GUT scale, because we obtain

$$DM_D\overline{D} \tag{6.4}$$

where D is the charge $-\frac{1}{3}$ quark in (4.3). D becomes heavy with the mass matrix M_D given by

$$M_D = V \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \tag{6.5}$$

where f_{ab} is symmetric. Note that $\mathrm{Det}M_D$ is in general nonzero. Thus, the above Yukawa coupling overcomes the first hurdle in the doublet-triplet splitting, removing the D and $\overline{D}$ particles.

Another contribution of the Yukawa coupling comes from picking up the same kind of trits from Ψ^a, Ψ^b, and Ψ_h. This gives mass to the Higgsino doublets

$$\tilde{H}_1 M_H \tilde{H}_2, \tag{6.6}$$

where we obtain the following 3×3 matrix for the three pairs,

$$M_H = V \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}, \tag{6.7}$$

showing that the mass matrix M_H is idential to the M_D. It is like introducing $\mathbf{5}_H\overline{\mathbf{5}}_H$ in the $SU(5)$ GUT. The flipped $SU(5)$ realizes the doublet-triplet splitting by excluding $\mathbf{5}_H\overline{\mathbf{5}}_H$ [20]. In our trits language, we cannot give such an assumption

because the Yukawa coupling contains both, as in the $SU(5)$ model. However, in our trits system we observed that the contributions come from two different kinds of trits combinations. Therefore, we have a room to introduce a new quantum number such that only different trits contribute in the Yukawa coupling.

We called this new quantum number *humor*. The **27** comes in three humors: $(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}), (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{3})$, and $(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$, forming the fundamental representation of a humor group. We may keep only the humor singlet component of the Yukawa couplings from Eq. (6.1). In this way, we can keep the Higgs doublet light, overcoming the second hurdle in the doublet-triplet splitting. However, we have not yet succeeded in picking up different humors among the Yukawa couplings in a natural way.

7. Conclusion

In this paper, we use the *trits* system $\{\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}\}$ to describe the maximaly broken(by three Wilson lines in the orbifold compactification) but maximally symmetric group among factor groups of the $E_8 \times E_8'$ heterotic string. We obtain the octa gauge group $SU(3)^8$ in two Z_3 orbifold compactifications and presented as Model A and Model B. These can be called octanification, the unification of all elementary particle forces in terms of eight $SU(3)$ factors. We presented all the matter spectrum in the $SU(3)$ trits terminology. Then, we searched for SSM vacua in two examples, SSM-I and SSM-II. Since the three Wilson line models render only one family, we have to remove one Wilson line to obtain three families. However, the vacuum with three Wilson lines is visionary in picking out two Wilson line models, and helps what happen in the removal of one Wilson line. This is a building-up approach after acquiring all the pieces. In this way, we observed an enhanced symmetry E_6 from $SU(3)^3$ and the physics behind this enhancement. We obtained three family supersymmetric standard models(SSM) with two Wilson lines. Also, we represented the low lying representations of E_6 and E_8 in terms of trits. This trits representation will make the study of exceptional groups as simple as that of the unitary groups.

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sector		state
UT		None
T0		3(1,1,1,3)(1,1,1,1) + ($\bar{3}$,3,1,1)(1,1,1,1) + (3,1,$\bar{3}$,1)(1,1,1,1) + (1,$\bar{3}$,3,1)(1,1,1,1)
T1	$(a_1;+)$	3(1,$\bar{3}$,1,1)(1,1,1,1) + ($\bar{3}$,1,1,$\bar{3}$)(1,1,1,1) + (3,1,3,1)(1,1,1,1) + (1,1,$\bar{3}$,3)(1,1,1,1)
T2	$(a_1;-)$	3(3,1,1,1)(1,1,1,1) + (1,$\bar{3}$,$\bar{3}$,1)(1,1,1,1) + (1,3,1,$\bar{3}$)(1,1,1,1) + (1,1,3,3)(1,1,1,1)
T3	$(a_3;+)$	(1,1,1,3)(1,1,1,3)
T4	$(a_3;-)$	(1,1,1,3)(1,1,1,$\bar{3}$)
T5	$(a_1,a_3;++)$	(1,$\bar{3}$,1,1)(1,1,1,3)
T6	$(a_1,a_3;+-)$	(1,$\bar{3}$,1,1)(1,1,1,$\bar{3}$)
T7	$(a_1,a_3;-+)$	(3,1,1,1)(1,1,1,3)
T8	$(a_1,a_3;--)$	(3,1,1,1)(1,1,1,$\bar{3}$)
T9	$(a_5;+)$	3(1,1,1,1)(1,1,$\bar{3}$,1) + (1,1,1,1)($\bar{3}$,1,1,3) + (1,1,1,1)(3,3,1,1) + (1,1,1,1)(1,$\bar{3}$,1,$\bar{3}$)
T10	$(a_5;-)$	(1,1,1,$\bar{3}$)(1,1,3,1)
T11	$(a_1,a_5;++)$	(1,1,$\bar{3}$,1)(1,1,$\bar{3}$,1)
T12	$(a_1,a_5;+-)$	($\bar{3}$,1,1,1)(1,1,3,1)
T13	$(a_1,a_5;-+)$	(1,1,3,1)(1,1,$\bar{3}$,1)
T14	$(a_1,a_5;--)$	(1,3,1,1)(1,1,3,1)
T15	$(a_3,a_5;++)$	3(1,1,1,1)(1,$\bar{3}$,1,1) + (1,1,1,1)(3,1,3,1) + (1,1,1,1)(1,1,$\bar{3}$,3) + (1,1,1,1)($\bar{3}$,1,1,$\bar{3}$)
T16	$(a_3,a_5;+-)$	(1,1,1,$\bar{3}$)(3,1,1,1)
T17	$(a_3,a_5;-+)$	3(1,1,1,1)($\bar{3}$,1,1,1) + (1,1,1,1)(1,3,3,1) + (1,1,1,1)(1,1,$\bar{3}$,$\bar{3}$) + (1,1,1,1)(1,$\bar{3}$,1,3)
T18	$(a_3,a_5;--)$	(1,1,1,$\bar{3}$)(1,3,1,1)
T19	$(+++)$	(1,1,$\bar{3}$,1)(1,$\bar{3}$,1,1)
T20	$(++-)$	($\bar{3}$,1,1,1)(3,1,1,1)
T21	$(+-+)$	(1,1,$\bar{3}$,1)($\bar{3}$,1,1,1)
T22	$(+--)$	($\bar{3}$,1,1,1)(1,3,1,1)
T23	$(-++)$	(1,1,3,1)(1,$\bar{3}$,1,1)
T24	$(-+-)$	(1,3,1,1)(3,1,1,1)
T25	$(--+)$	(1,1,3,1)($\bar{3}$,1,1,1)
T26	$(---)$	(1,3,1,1)(1,3,1,1)

Table 2: No spectrum in the untwisted sector. The model is $v = (0^5\ \frac{1}{3}\ \frac{1}{3}\ \frac{2}{3})(0^8)$, $a_1 = (\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\ 0\ 0\ \frac{1}{3}\ \frac{1}{3}\ \frac{5}{3})(0^8)$, $a_3 = (0^8)(0\ 0\ 0\ 0\ 0\ \frac{1}{3}\ \frac{1}{3}\ \frac{2}{3})$, $a_5 = (0\ 0\ 0\ 0\ 0\ \frac{2}{3}\ \frac{2}{3}\ \frac{4}{3})(\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\ 0\ 0\ \frac{1}{3}\ \frac{1}{3}\ \frac{5}{3})$

sector		state
U		$3(\bar{3},3,1,\bar{3})(1,1,1,1)$
T0		$3(1,1,1,3)(1,1,1,1) + (\bar{3},3,1,1)(1,1,1,1)$
		$+ (3,1,\bar{3},1)(1,1,1,1) + (1,\bar{3},3,1)(1,1,1,1)$
T1	$(a_1;+)$	$3(1,\bar{3},1,1)(1,1,1,1) + (\bar{3},1,1,\bar{3})(1,1,1,1)$
		$+ (3,1,3,1)(1,1,1,1) + (1,1,3,3)(1,1,1,1)$
T2	$(a_1;-)$	$3(3,1,1,1)(1,1,1,1) + (1,\bar{3},\bar{3},1)(1,1,1,1)$
		$+ (1,3,1,3)(1,1,1,1) + (1,1,\bar{3},\bar{3})(1,1,1,1)$
T3	$(a_3;+)$	$(1,1,1,3)(1,1,1,3)$
T4	$(a_3;-)$	$(1,1,1,3)(1,1,1,\bar{3})$
T5	$(a_5;+)$	$(1,1,1,3)(1,1,3,1)$
T6	$(a_5;-)$	$(1,1,1,3)(1,1,\bar{3},1)$
T7	$(a_1,a_3;++)$	$(1,\bar{3},1,1)(1,1,1,3)$
T8	$(a_1,a_3;+-)$	$(1,\bar{3},1,1)(1,1,1,\bar{3})$
T9	$(a_1,a_5;++)$	$(1,\bar{3},1,1)(1,1,3,1)$
T10	$(a_1,a_5;+-)$	$(1,\bar{3},1,1)(1,1,\bar{3},1)$
T11	$(a_1,a_3;-+)$	$(3,1,1,1)(1,1,1,3)$
T12	$(a_1,a_3;--)$	$(3,1,1,1)(1,1,1,\bar{3})$
T13	$(a_1,a_5;-+)$	$(3,1,1,1)(1,1,3,1)$
T14	$(a_1,a_5;--)$	$(3,1,1,1)(1,1,\bar{3},1)$
T15	$(a_3,a_5;++)$	$(1,1,1,3)(1,\bar{3},1,1)$
T16	$(a_3,a_5;+-)$	$(1,1,1,3)(3,1,1,1)$
T17	$(a_3,a_5;-+)$	$(1,1,1,3)(\bar{3},1,1,1)$
T18	$(a_3,a_5;--)$	$(1,1,1,3)(1,3,1,1)$
T19	$(+++)$	$(1,\bar{3},1,1)(1,\bar{3},1,1)$
T20	$(++-)$	$(1,\bar{3},1,1)(3,1,1,1)$
T21	$(+-+)$	$(1,\bar{3},1,1)(\bar{3},1,1,1)$
T22	$(+--)$	$(1,\bar{3},1,1)(1,3,1,1)$
T23	$(-++)$	$(3,1,1,1)(1,\bar{3},1,1)$
T24	$(-+-)$	$(3,1,1,1)(3,1,1,1)$
T25	$(--+)$	$(3,1,1,1)(\bar{3},1,1,1)$
T26	$(---)$	$(3,1,1,1)(1,3,1,1)$

Table 3: Opposite chirality is written in the untwisted sector. The model is $v = (0^5\ \frac{1}{3}\ \frac{1}{3}\ \frac{2}{3})(0^8)$, $a_1 = (\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\ 0\ 0\ \frac{1}{3}\ \frac{1}{3}\ \frac{5}{3})(0^8)$, $a_3 = (0^8)(0\ 0\ 0\ 0\ 0\ \frac{1}{3}\ \frac{1}{3}\ \frac{2}{3})$, $a_5 = (0^8)(\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\ 0\ 0\ \frac{1}{3}\ \frac{1}{3}\ \frac{5}{3})$.

sector	in trits	in $(E_6, SU(3)_4)(SU(3)'_5, SU(3)'_6, SU(3)'_7, SU(3)'_8)$
UT	None	None
T0	9(1,1,1,3)(1,1,1,1) $3(\bar{3}$,3,1,1)(1,1,1,1) $3(3,1,\bar{3}$,1)(1,1,1,1) $3(1,\bar{3}$,3,1)(1,1,1,1)	$9(\mathbf{1},\mathbf{3})(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$ $3(\mathbf{27},\mathbf{1})(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
T3	3(1,1,1,3)(1,1,1,3)	$3(\mathbf{1},\mathbf{3})(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$
T4	$3(1,1,1,3)(1,1,1,\bar{3})$	$3(\mathbf{1},\mathbf{3})(\mathbf{1},\mathbf{1},\mathbf{1},\bar{\mathbf{3}})$
T9	$9(1,1,1,1)(1,1,\bar{3},1)$ $3(1,1,1,1)(\bar{3},1,1,3)$ 3(1,1,1,1)(3,3,1,1) $3(1,1,1,1)(1,\bar{3},1,\bar{3})$	$9(\mathbf{1},\mathbf{1})(\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{1})$ $3(\mathbf{1},\mathbf{1})(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{3})$ $3(\mathbf{1},\mathbf{1})(\mathbf{3},\mathbf{3},\mathbf{1},\mathbf{1})$ $3(\mathbf{1},\mathbf{1})(\mathbf{1},\bar{\mathbf{3}},\mathbf{1},\bar{\mathbf{3}})$
T10	$3(1,1,1,\bar{3})(1,1,3,1)$	$3(\mathbf{1},\bar{\mathbf{3}})(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})$
T15	$9(1,1,1,1)(1,\bar{3},1,1)$ $3(1,1,1,1)(\bar{3},1,1,\bar{3})$ 3(1,1,1,1)(3,1,3,1) $3(1,1,1,1)(1,1,\bar{3},3)$	$9(\mathbf{1},\mathbf{1})(\mathbf{1},\bar{\mathbf{3}},\mathbf{1},\mathbf{1})$ $3(\mathbf{1},\mathbf{1})(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\bar{\mathbf{3}})$ $3(\mathbf{1},\mathbf{1})(\mathbf{3},\mathbf{1},\mathbf{3},\mathbf{1})$ $3(\mathbf{1},\mathbf{1})(\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{3})$
T16	$3(1,1,1,\bar{3})(3,1,1,1)$	$3(\mathbf{1},\bar{\mathbf{3}})(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$
T17	$9(1,1,1,1)(\bar{3},1,1,1)$ 3(1,1,1,1)(1,3,3,1) $3(1,1,1,1)(1,1,\bar{3},\bar{3})$ $3(1,1,1,1)(1,\bar{3},1,3)$	$9(\mathbf{1},\mathbf{1})(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$ $3(\mathbf{1},\mathbf{1})(\mathbf{1},\mathbf{3},\mathbf{3},\mathbf{1})$ $3(\mathbf{1},\mathbf{1})(\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\bar{\mathbf{3}})$ $3(\mathbf{1},\mathbf{1})(\mathbf{1},\bar{\mathbf{3}},\mathbf{1},\mathbf{3})$
T18	$3\ (1,1,1,\bar{3})(1,3,1,1)$	$3(\mathbf{1},\bar{\mathbf{3}})(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$

Table 4: SSM-I: The shift vector and Wilson lines are $v = \frac{1}{3}(0^5\ 1\ 1\ 2)(0^8), a_3 = \frac{1}{3}(0^8)(0^5\ 1\ 1\ 2), a_5 = \frac{1}{3}(0^5\ 2\ 2\ 4)(1\ 1\ 1\ 0\ 0\ 1\ 1\ 5)$

sector	in trits	in $(SU(3)_1, SU(3)_2, SU(3)_3, SU(3)_4)(E_6', SU(3)_7')$
UT	None	None
T0	$9(1,1,1,3)(1,1,1,1)$	$9\mathbf{(1,1,1,3)(1,1)}$
	$3(\bar{3},3,1,1)(1,1,1,1)$	$3(\bar{\mathbf{3}},\mathbf{3},\mathbf{1},\mathbf{1})(\mathbf{1},\mathbf{1})$
	$3(3,1,\bar{3},1)(1,1,1,1)$	$3(\mathbf{3},\mathbf{1},\bar{\mathbf{3}},\mathbf{1})(\mathbf{1},\mathbf{1})$
	$3(1,\bar{3},3,1)(1,1,1,1)$	$3(\mathbf{1},\bar{\mathbf{3}},\mathbf{3},\mathbf{1})(\mathbf{1},\mathbf{1})$
T1	$9(1,\bar{3},1,1)(1,1,1,1)$	$9(\mathbf{1},\bar{\mathbf{3}},\mathbf{1},\mathbf{1})(\mathbf{1},\mathbf{1})$
	$3(\bar{3},1,1,\bar{3})(1,1,1,1)$	$3(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\bar{\mathbf{3}})(\mathbf{1},\mathbf{1}))$
	$3(3,1,3,1)(1,1,1,1)$	$3(\mathbf{3},\mathbf{1},\mathbf{3},\mathbf{1})(\mathbf{1},\mathbf{1})$
	$3(1,1,\bar{3},3)(1,1,1,1)$	$3(\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{3})(\mathbf{1},\mathbf{1})$
T2	$9(3,1,1,1)(1,1,1,1)$	$9(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})(\mathbf{1},\mathbf{1})$
	$3(1,\bar{3},\bar{3},1,)(1,1,1,1)$	$3(\mathbf{1},\bar{\mathbf{3}},\bar{\mathbf{3}},\mathbf{1})(\mathbf{1},\mathbf{1})$
	$3(1,3,1,\bar{3})(1,1,1,1)$	$3(\mathbf{1},\mathbf{3},\mathbf{1},\bar{\mathbf{3}})(\mathbf{1},\mathbf{1})$
	$3(1,1,3,3)(1,1,1,1)$	$3(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{3})(\mathbf{1},\mathbf{1})$
T9	$9(1,1,1,1)(1,1,\bar{3},1)$	$9(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})(\mathbf{1},\bar{\mathbf{3}})$
	$3(1,1,1,1)(\bar{3},1,1,3)$	
	$3(1,1,1,1)(3,3,1,1)$	$3(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})(\overline{\mathbf{27}},\mathbf{1})$
	$3(1,1,1,1)(1,\bar{3},1,\bar{3})$	
T10	$3(1,1,1,\bar{3})(1,1,3,1)$	$3(\mathbf{1},\mathbf{1},\mathbf{1},\bar{\mathbf{3}})(\mathbf{1},\mathbf{3})$
T11	$3(1,1,\bar{3},1)(1,1,\bar{3},1)$	$3(\mathbf{1},\mathbf{1},\bar{\mathbf{3}},\mathbf{1})(\mathbf{1},\bar{\mathbf{3}})$
T12	$3(\bar{3},1,1,1)(1,1,3,1)$	$3(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})(\mathbf{1},\mathbf{3})$
T13	$3(1,1,3,1)(1,1,\bar{3},1)$	$3(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})(\mathbf{1},\bar{\mathbf{3}})$
T14	$3(1,3,1,1)(1,1,3,1)$	$3(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})(\mathbf{1},\mathbf{3})$

Table 5: SSM-II: The shift vector and Wilson lines are $v = \frac{1}{3}(0^5\ 1\ 1\ 2)(0^8), a_1 = \frac{1}{3}(1\ 1\ 1\ 0\ 0\ 1\ 1\ 5)(0^8), a_5 = \frac{1}{3}(0^5\ 2\ 2\ 4)(1\ 1\ 1\ 0\ 0\ 1\ 1\ 5)$